THE EISENHOWER LIBRARY
3 1151 02807 9964

98.091

Library

of the
Johns Hopkins University

JAV THESIS
Rabinovitch, I. E.
1899
C. 2

The Foundations of the Euclidian Geometry, as viewed from the Standpoint of Kinematics.

Introduction.

Both mathematicians and philosophers at present agree that, although the science of mathematics as a whole is undoubtedly the most exact of sciences, one of her most important branches, at once the oldest and the most fascinating one, namely geometry, has to some extent lost in its prestige, and can no longer be posed by epistemology as the prototype of purely deductive, a priori science, and as a proof of the existence of certain innate ideas, having a purely transcendental origin, wholly independent of experience and partly conditioning it. This change in the view upon geometry has taken place during the past century of all-pervading doubt and criticism and strange to say, it did not originate with men outside of the mathematical profession but with those who had the greatest interest in preserving her sanctity in keeping up the halo of her alleged transcendental origin. The very priests who worship at her shrine the greatest mathematicians who contributed most to her fabulous growth and development in the 19th century — Gauss, Riemann

Helmholtz, Beltrami, Clifford among the immortal dead, and many prominent names still among us, have done most to cause this change. At present it is almost regarded as a heresy to attempt to restore some of the old prestige to the science which was considered by the Greeks to be the prototype of all science and all philosophy.

All this change of view has occurred, of course not with regard to the method employed in geometry the soundness and legitimacy of which have never been seriously doubted but with regard to the very foundations upon which geometry rests. It is the body of axioms and postulates, the definitions and common notions, — propositions and assumptions, both implicit and explicit, which had for a very long time been considered as self-evident, intuitive and independent of all elaborate proof — propositions that need only be stated in order to elicit unconditional consent, - it is this body of so called self-evident truths which at present are questioned and doubted and by many relegated to the realm of empirical true rules with a certain degree of approximation and capable

of being modified in an infinity of ways and, hence, of giving rise to a corresponding multitude of geometries, each consistent in itself but in contradiction with the others, each as perfect in theory as any other, and all very nearly agreeing with our limited experience.*) It is, however, admitted on all sides that the old system, namely the Euclidian system, more than all others seems to agree with the results of our experience, as far as this last goes, and if we were able to extend our experience considerably beyond the limits of the [illegible] and [illegible] then we should find its norms to remain unaltered and not needing revision, it would to a certain extent prove the physical reality of the Euclidian geometry and the unreality of the other systems although even then the others would still be theoretically admissible and would form a body of imaginary geometries

Now, this is rather a peculiar state of the geometrical science, singular in its kind. For, while in other branches of science two contradictory systems of thought would readily be allowed to stand side by side,

*) Professor F. Klein [illegible] in his memoirs "Über die sogenannte Nicht-Euklidische Geometrie", Math. Ann. Bd IV, VI, XXXVII and in his "Nicht Euklidische Geometrie", lithographirte Ausgabe, Gött., 1893, forcibly presents & defends this opinion. See lithogr. Lectures I, pp. 298–365, Math. Ann. XXXVII, p. 570.

both claiming to represent the truth simultaneously, - while, for instance, the Ptolemian and Copernican systems of Astronomy could not avowedly coexist - the latter having superseded the former, as soon as it was found better to agree with astronomical observations and with the abstract laws of mechanics, - while no quarter was given to the Aristotelian theory of the fixity of species by the new evolutionary systems of Lamarck and Darwin, or to the ancient doctrine of the four elements by modern chemistry and physics - the contradictory systems of geometry according to some mathematicians of note would be allowed to stand together, side by side, and be of equal theoretical (if not practical) value and importance. So, for instance, F. Klein in his memoirs in the Math. Ann., Vol's 4, 6, 7 and 37 and in his "Nicht Euklidische Geometrie", second impression, Göttingen, 1893, develops from the projective point of view three systems of geometry - the Elliptic, the Hyperbolic and the Parabolic systems (which in broad features had been drawn already by Riemann), corresponding to the three possible

hypotheses which can be made concerning our space, namely, as possessing constant positive, negative or zero curvature. Giving no theoretical preference to any of these systems, he even goes so far as to think that the question as to, whether one of these systems is to be preferred as expressing the real relations of our space, is unanswerable, since on allowing the radius of curvature to be sufficiently great the elliptic or hyperbolic geometry would give results approximating with as great a degree of accuracy as we please to the results obtainable by the most exact measurements, performed with the most powerful telescopes upon distances such as are involved in the determination of the annual parallax of a fixed star. He prefers the parabolic geometry, however, on account of its presenting the simplest hypothesis in the theory of measurement. So he says in his lectures on the Non-Euclidian geometry referred to above, first part, page 277*) — "There is, however, on the other hand, no lack of enthusiasts, who do not answer the question, in the way we have done, by asserting that to our

*) See also pp. 161-170, where this idea is presented with greater force and elegance.

conception and experience of space could with sufficient precision alike correspond the hyperbolic or elliptic, as well as the parabolic system of measurement, and that we decide in favor of the parabolic solely on account of its offering the simplest hypothesis (just as in physics among hypotheses of equal probability the simplest is always allowed to prevail)." — Each of these geometries, according to Klein in another place of same work (p. 95), admits of an infinity of different space-forms, — in which respect he differs from Killing, who thinks that only in case of the elliptic geometry an infinite number of space forms Raumformen- is possible. For says "To thus expressly contradict the remark of Killing that in case of the hyperbolic or parabolic metrics there exists the possibility only for one space-form, we say on the contrary that also in these cases there exists an infinite number of space forms." It has come to pass, indeed, that some mathematicians are vying with one another in devising new space forms as they call them, for which new systems of geometry are supposed to hold. Clifford Klein,

Lindemann, Killing and some others have contributed much to this field of investigation. An enumeration and description of some of these geometries would lead us too far, and the reader interested is referred to the numerous memoirs in the Math. Ann. and Crelle's Journal and to separate books and reprints from mathematical and philosophical periodicals which have appeared since the beginning the [illegible] — this up to the end of the past century.

It may however be said without exaggeration that most of these space forms rather impress the reader with the ingenuity of their inventors than with their actual value in bearing upon the question of the foundations of geometry. There seems to be rather too much license given to the imaginative faculty of the human mind; and while the origin of almost all investigations of this nature is to be sought in the importance given to the Non Euclidian geometry by the deep searching criticism of Riemann, inaugurated by his thesis "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" with Werke pp. 254. 269. — It was not intended for publication by the author in the form it appeared after his death in the Göttinger Abhandlungen S. a. Nr. 13 Euclid I p. 216. Le "transformationsgruppen" III pp 105-486.

formulated postulates without giving, [illegible] answers to some of them — the new systems cannot handle even very many theorems with them and it may be stated as a certainty that many of them would not stand a scrutinizing criticism and would have to be relegated to the realm of fancy rather than be classed with such an exact science as mathematics. To quote an instance the elliptic space, i.e. one of positive curvature, with two geodesics meeting only in one pt., described by Klein, Lindemann[1] and Killing[2] is one of such systems. Beltrami in his "Teoria fondamentale degli Spatii di Curvatura Constante"[3] makes the expres statement that any two geodesics in a space of positive curvature meet in two antipodal pts., through which a whole pencil of (∞ infinity) similar geodesics must pass, and to this statement Klein takes exception in his "Nicht Euklid. Geometrie" pp. 240-242 and in other places of the same work, in the Math. Ann. t. 6, p. 125 and t 37, p 554 et seq. — Another instance is the spiral space-form in which a rotation of a rigid body is accompanied by an increase or decrease in its volume, so that by a continuous rotation about a fixed point, any

1) Neuere Untersuchungen über Geometrie p. II.
2) Crelle t. 98 p. 72
3) Annali di Matematica ser. 2, II (1868. The French translation of this work of Beltrami and also of his famous Saggio, by Houël appeared in the Jour. de [illegible]

arbitrarily chosen body can be made to enclose any given point of space, and by an inverse rotation the body can be made to shrink down to an arbitrary small portion of space around the fixed point. (Killing: Ueber die Grundlagen der Geometrie. Crelle's Journ., t. 109, 1891, pp. 185–266).

There are however other mathematicians who, agreeing in the main that the foundations of geometry have thus far not been laid down with any degree of certitude and that they are, therefore, open to considerable differences of opinion, think nevertheless that there ought to be some objective truth concerning the nature of these foundations, and that it is not at all unlikely that some day a satisfactory body of axioms and postulates may be found, which will prove undebatable. The question is only in finding the minimum of simple truths, derived from experience as an original source and formulated by abstraction into a body of definitions and propositions, which on account of their uncontestable efficiency as a basis for geometry, on the one hand, and by their unquestionable reality on the other hand, as well as by their being

irreducible to a smaller number with equal efficiency, should carry conviction into the mind of the mathematician, whose taste is especially just down — this relation and [illegible] satisfy him that the basis is a unified whole, without leaks, and that it is capable to stand the test of a search for skepticism. Of course not a metaphysical skepticism putting questions of the nature of whether space and time or even matter and mind, the Ego, the Universe, and so on, have any reality, objective or subjective, phenomenal or noumenal or? Among these mathematicians is especially to be mentioned Sophus Lie, who, it seems, has contributed more than any other contemporary mathematician, to sound views in this matter, by his treating the so called Riemann Helmholtz problem in a masterly way which won for him the Lobatchevski Prize in 1897. So Lie says in his Transformationsgruppen, vol. III, p. 398, "We wish, however, to express the opinion which is a conviction with us [illegible] is unsere Ueberzeugung die Auffassung aussprechen, dass es in no wise impossible to establish a system of

geometrical axioms which shall be sufficient and shall contain nothing superfluous. It is [illegible] certain however, that there are very few investigations which have actually furthered the problem as to the foundations of geometry."

In another place of the same work (p. 536) he says:

"Geometry in its different stages ought, as much as possible, to be founded on a purely geometrical basis; this is a demand with which everybody, undoubtedly, will agree."

"For the first stage of geometry are, in the first place necessary certain fundamental conceptions, like space, curve, point and surface, second come certain axioms concerning for instance, the properties of the right line, the existence of a plane and so on. Every new conceivable stage is characterized by the introduction of new axioms, one stage, for inst., by the axiom of parallels, another by the Cantor-axiom." — In my own work this last axiom, i.e. that the str. line is a number-manifold, will be proved one of the fundamental properties of the str. line, from which its construction and all its other properties

are obtained — "Upon this axiom it is possible to establish rationally the conceptions of area, length of arc &c, while Euclid virtually needs a separate axiom in each case."

"The great question is now, what axioms on each stage are not only sufficient, but also necessary, in other words, are indispensable. In the answer to this question the whole problem of the foundations of Geometry would find its solution."

And then further (p. 537) Lie sketches a programme for the mathematician who would undertake to establish the necessary axioms, which perhaps might prove fewer than those which he assumed for the purpose of solving the Riemann-Helmholtz problem — "First one would have to establish certain fundamental conceptions, like space, point, curve, surface, and also the conception of motion..."

"As a first axiom one would have to establish the following: If a point P is fixed, every other point can still describe a surface which does not pass through the point P. In this may be found the reason that two points in

all rigid motions (Bewegungen) remain separated." — These two conceptions are certainly also connected in my treatment (see definition of distance and the following proof of the existence of a sphere) except that the second is coming first as the simplest one and following at once from the conception of rigidity.—

"As a second axiom it would be necessary to assume that — When two points P_1 and P_2 are fixed there are still an infinity of other points which remain fixed simultaneously and these points form one and only one straight line passing through P_1 and P_2." — I think Lie would certainly not object to a proof of this proposition which at once makes space a number-manifold and thus supplies also the Cantor-axiom. He says that these propositions are not yet sufficient for the first stage of geometry. Now, I think that by means of only one additional axiom (viz. axiom 1, p.) which, according to my mind happens to coincide with Lie's fundamental notion of a continuous group of displacements, I have succeeded in establishing all that is necessary

for the first stage together with the most important of the second postulate stage, namely the parallel postulate, in a sense, however, that does not exclude the spherical and pseudospherical geometries proving only the necessity of a plane geometry in the Euclidean sense.—

Lie himself treated this subject from the point of view of continuous group-transformations. Starting with the Riemann Helmholtz postulate that space is a manifold of three dimensions, in which the position of the single element (the point) is determined by three coordinates, and adding a few very simple postulates, characterizing the group of continuous motions of rigid bodies (i.e. transformations in which every two pts. have one essential invariant),— he showed that there remains only the possibility for the Euclidian and the two Non-Euclidian systems of motions.[1] The two last ones are such as leave invariant respectively the imaginary surface $x_1^2 + x_2^2 + x_3^2 + 1 = 0$ (Riemannian group), or the real surface $x_1^2 + x_2^2 + x_3^2 - 1 = 0$ (Lobatchevsky

1) Transformationsgruppen, III, pp. 404, 479 (the [illegible] the fifth Chapter is devoted to the Riemann-Helmholtz problem. See also Leipziger Berichte, 1890, pp. 356–418, 284–321, and 1892, pp. 297–305.

groups of motions) to groups of Euclidean motions together with the groups of transformations of similar figures, are characterized by their leaving invariant the absolute,

$$X^2 + Y^2 + Z^2 = 0 \quad , \quad T = 0.$$ [1]

The following are the axioms [2] which Lie considers sufficient for his investigation:

I) Space is a number-manifold of three dimensions, R_3

II) The displacements or motions of R_3 form a real continuous group of point transformations.

III) If any real pt. of general position, y_1^0, y_2^0, y_3^0 is held fixed, all the real points x_1, x_2, x_3, into which another real point x_1^0, x_2^0, x_3^0 can be moved, satisfy an equation with real coefficients, of the form

$$W(y_1^0, y_2^0, y_3^0; x_1^0, x_2^0, x_3^0; x_1, x_2, x_3) = 0,$$

which is not fulfilled for $x_1 = y_1^0$, $x_2 = y_2^0$, $x_3 = y_3^0$, and which, in general represents a real surface, passing through x_1^0, x_2^0, x_3^0. (A synthetic proof of this proposition is given in theorem I in my present Dissertation).

IV) About the point y_1^0, y_2^0, y_3^0 a finite triply-extended

1) Transformationsgruppen, III, p. 218

2) Ibid. pp. 506-507.

region may be so bounded off that after passing the point x_1^0, x_2^0, x_3^0, every other real point x_1, x_2, x_3 of the region can still pass by continuous motion into the position of every other real point of the region which satisfies the equation $W=0$, and which is joined to the point x_1^0, x_2^0, x_3^0 by an irreducible continuous series of points. (This proposition is also proved in the theorem referred to above)

In the Leipziger Berichte, 1890, pp. 357-358, he characterizes the axioms necessary and sufficient to define the Euclidean and the two Non-Euclidean motions in a somewhat different way which however amounts to the same thing. His analytical formulae are interesting, and I repeat them here. The assumptions are:

The infinite aggregate of real transformations of the form of R_3 x, y, z, is given by the equations: $x_1 = \chi(x, y, z, a_1, a_2 \ldots)$ $y_1 = \varphi(x, y, z, a_1, a_2 \ldots)$ $z_1 = \psi(x, y, z, a_1, a_2 \ldots)$. These equations have to satisfy the following conditions:

A) The functions χ, φ, ψ are analytical fns. of the coordinates x, y, z and of the parameters $a_1, a_2 \ldots a_r$.

B) Any two pts. x_1, y_1, z_1; x_2, y_2, z_2 have one essential invariant under all transformations of the group, of the form $\Omega(x_1, y_1, z_1; x_2, y_2, z_2) = \text{const}$. Hence $\Omega(x_1, y_1, z_1; x_2, y_2, z_2) = \Omega(x_1', y_1', z_1'; x_2', y_2', z_2')$, where x_1', y_1', z_1'; x_2', y_2', z_2' are the new positions of the original pair of pts. x_1, y_1, z_1; x_2, y_2, z_2 which these obtain in virtue of any transformation of the group.

C) The group is transitive, i.e. any point of the R_3 can be transformed into any other point. If, however, the point x_1, y_1, z_1 is fixed [illegible] point x_2, y_2, z_2 can assume ∞^2 different positions, which are defined by the equation: $\Omega(x_1, y_1, z_1; x_2', y_2', z_2') = \Omega(x_1, y_1, z_1; x_2, y_2, z_2)$. If two points x_1, y_1, z_1 and x_2, y_2, z_2 are fixed, any third point [illegible] general position x_3, y_3, z_3 can assume ∞^1 different positions x_3', y_3', z_3' defined by the two equations:

$$\Omega(x_1, y_1, z_1; x_3', y_3', z_3') = \Omega(x_1, y_1, z_1; x_3, y_3, z_3)$$

$$\Omega(x_2, y_2, z_2; x_3', y_3', z_3') = \Omega(x_2, y_2, z_2; x_3, y_3, z_3).$$

The point x_3, y_3, z_3 must be of general position, in order to be able to assume ∞^1 different positions, since there

exists a singly-infinite number of points defined by the last two equations ∩, such that $x_3', y_3', z_3' = x_3, y_3, z_3$, namely the ∞^1 points collinear with $x_1, y_1, z_1; x_2, y_2, z_2$.—

If three points $x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3$ are fixed all points of space remain fixed, and three similar equations will be satisfied only for $x_4' = x_4, y_4' = y_4, z_4' = z_4$.

The interpretation of the results obtained by Lie with regard to the possibility of the two groups of Non-Euclidian motions comes, I think, to mind, represents a well solved mathematical problem in complete accord with the interpretation of similar results obtained by him for n-dimensional manifolds. A concrete interpretation of these must be found in our empirical space, for which the Euclidian axioms hold, in order to appreciate their full geometrical significance.

Not considering myself competent at present to undertake even a partial solution of this problem, I wish, however, to indicate that for two dimensional manifolds an interpretation seems to lie near at hand, not at

all in conflict with our Euclidian conceptions of space. The Non-Euclidian groups of transformations, assuming an invariant between two elements only of the most general kind and preserving projective relations to certain special forms of the fundamental quadric must represent the real metric relations of our space as imaged by projection upon certain surfaces of the second degree. In such a projective image the anharmonic ratio of four points or some function of it will remain unaltered which will have to be taken for a definition of distance or angle in these transformed metrics so that these metrics will coincide with the non-Euclidian metrics developed by Klein. For an interpretation of this nature, we may refer to Poincaré's paper on the foundations of geometry in the Bull. de la Soc. de France, t. 16, Nov. 1887. Another interpretation of these groups might be found in the metrics upon surfaces of positive and negative curvature

when the str. lines are replaced by geodesics on these surfaces, an interpretation which has been fully justified by the works of Beltrami, to which reference will be made later.

Klein in his "Nicht-Euklid. Geom." has shown, it seems to me conclusively (although he intended his procedure to illustrate and to show in a concrete manner the possibility of the plane having elliptic or hyperbolic metrics) that what he calls an elliptic plane is actually the central projection upon a Euclidian plane of the metrics upon a sphere*), and what he calls the hyperbolic plane; the orthographic projection of a system of metrics upon a sphere touching the plane in which str. lines are represented by circles orthogonal to the equator of the sphere which is parallel to the plane. At any rate this also shows that by certain processes of projection and by making certain conventions as to the meaning of "distance" and "angle" we may be able to account for the two Non-Euclidian groups of motions, as well as for the Non-Euclidian metrics

*) The radius of the sphere is taken to be $-2k$, and $-4k^2$ is the measure of curvature of the elliptic and hyperbolic plane, resp. See "Nicht-Euklid." pp. 94-97 and pp 220 - 237.

deduced by Klein from the Cayleyan metrics. The Non-Euclidian groups of rotation also however for three dimensions, as well as the Euclidian and the Non-Euclidian groups of displacements for manifolds of a higher number of dimensions than three, which Lie derives from postulates similar to those he assumed for R_3, still need an interpretation, and this interpretation seems to lie in the change of element from point to a figure depending upon any number of parameters, which is Plücker's idea of making our space n-dimensional (To this I shall yet have occasion to refer later)—

Another eminent thinker on the subject who, in the main, holds the same opinions and who has written some expositions of the ideas of Lie and of his own views on this subject, is the illustrious French mathematician Poincaré. The first publication of his on this subject is the paper in the Bull. de la Soc. de France quoted above; then papers of his on the same subject appeared in the Revue Générale des Sciences pures et

t. III 1892, and in the Revue de Metaphysique et de Morale. Another paper, in which he further develops and complements his views in the previous papers, and which has been translated into English by McCormack appeared in the Monist, vol. IX, 1898. I quote extensively from this paper, as I find in it so many points of agreement with some of my own views about the relation of experience and pure reasoning to the formation of our geometrical notions, also in relation to the number of dimensions of space and other points:

He begins thus, "Our sensations cannot give us the notion of space. That notion is built up by the mind from elements pre-existing in it, and external experience is simply the occasion for its exercising this power..."

He maintains further that variations in our sensations give rise to our notions of space. We observe two kinds of changes in our impressions, which we thus separate into two classes:

1) External changes independent of our will.

2) Internal changes accompanied by voluntary muscular exertions.

The external changes [illegible] two subdivisions:

1) Displacements capable of being corrected by an internal change.

2) Alterations or physical changes, not having this property.

Only displacements are the object of geometry.

An identical displacement can be repeated a number of times. Hence the introduction of number.

The ensemble, or aggregate of displacements form a group, since the combination of any number of these is one of the aggregate.

The notion of group could not be formed by a priori reasoning, but by experience together with reasoning. We abstract from the concrete alterations which may accompany displacements, so that geometry is safe from all revision.

[illegible] experience teaches us that a certain phenomenon

does not correspond to the laws of the group, we strike it from the list of displacements. When it obeys these laws only approximately, we consider the change by an artificial convention as the resultant of two compound changes. One is regarded as a displacement, rigorously satisfying the laws of the group, while the second is regarded as a qualification or addition. Thus we say that solids undergo not only great changes of position, but also small flexions and small thermal alterations." (Compare with this Postulate 1 and Scholium to Definition 8) in my Dissertation).

"The fact that the displacements form a group contains in a germ a host of important consequences. Space must be homogeneous; that is all points are capable of playing the same part..."

"Being homogeneous it will be unlimited; for a category that is limited cannot be homogeneous, seeing that the boundaries cannot play the same part as the centre. But this does not say that it is infinite,

for a sphere is an unbounded surface; and yet it is finite.

To this reasoning I should object, for I should ask: Is not a sphere a bounded body, and therefore non-homogeneous in the third dimension? If space were finite it would not be homogeneous in some dimension, but as all dimensions belong to space, it would be non-homogeneous in some of those dimensions, hence we could not say without limitation that any two displacements form a new displacement.

In my proof of the infinite extent of the str. line I do not, however, assume the infinity of space. I only postulate that "each point is capable of playing the same part as any other", which is also postulated by Poincaré as being involved in the notion of the group. Hence, from any point we can describe a sphere with a distance as radius given by a previous consideration, if it is possible to do it once at least, and for one given distance. (See Theorem 2).

After positing a continuum, or a group of displacements, and defining what is meant by subgroups, isomorphism, invariant subgroups &c. into which I cannot go in detail without transcribing the paper as a whole, Poincaré goes on to say that by experience combined with abstraction we arrive at the notion of a rotative subgroup, or the ensemble of displacements which conserve a certain system of sensations. Then he says,

"By new experiences, always very crude, it is then shown:

(1) That any two rotative subgroups have common displacements.

(2) That the [illegible] common displacements all interchangeable among one another, form a sheaf, which may be called a rotative sheaf. (Rotations about a fixed axis).

(3) That any rotative sheaf forms part not only of two rotative subgroups but of an infinity of them. There is the origin of the notion of the str. line as the rotative subgroup was the origin of the notion

of the point."

In these few sentences one might discover a somewhat crude empirical statement of the facts which I availed myself in constructing the straight line in Theorem 2.

He then goes on to say that the existence of an invariant subgroup, namely the subgroup of translations, in which all displacements are interchangeable, is the fact that determines our choice in favor of Euclid as against that of Lobatchevsky because the group that corresponds to the geometry of Lobatchevsky does not contain such an invariant subgroup."

When he comes to the discussion of dimensions, Poincaré points out the distinction, from the point of view of the theory of groups, between the order k and the degree n of a group, and states that the order k is the more important characteristic of a group. So that two groups can be isomorphic (i.e. their operations obey the same laws of com-

bination and hence have the same number of subgroups, invariant subgroups &c.) and still in a different degree provided their order k is the same. In continuous groups, in general, and in the groups of displacements, in particular, the object of operations "is the ensemble of a certain number n of quantities, susceptible of being varied in a continuous manner, which quantities are called coordinates." "Then, every infinitesimal operation of the group can be decomposed into k infinitesimal operations belonging to k given sheaves. The number n of the coordinates (or of the dimensions) is then the degree, and the number k of the components of an infinitesimal operation is the order." "The degree is an element intrinsically material and secondary and the order a formal element" - The study of the groups is mainly the study of its formal properties. The order r corresponds to the number of essential parameters of a group of transformations

"The groups of displacements is of the sixth order."
- The order k in case of R_3 is 6, since R_3 can have ∞^6 displacements. - its to the degree n - ℓ depends upon the choice of element:

If we choose the different transformations of a rotative subgroup, we get a triple infinity of elements. "The degree of the group is three. We have chosen the point as the element of space and given to space three dimensions.

"Choosing the different transformations of a helicoidal subgroup, we obtain a quadruple infinity of elements. We have chosen the str line as the element of space which gives to space four dimensions."

"Suppose finally that we choose the different transformations of a rotative sheaf. The degree would then be five. We have chosen as the element of space the figure formed by a str. line and a point on that str. line. Space would have

five dimensions."

The introduction of a group more or less complicated appears to be absolutely necessary. Every purely statical theory of the number of dimensions will give rise to many difficulties, and it will always be necessary to fall back upon a dynamical theory.—

I wish to observe here that the deduction of the number of dimensions in my Dissertation is based upon kinematical principles.—

When I pronounce the word length, a word which we frequently do not think necessary to define, I implicitly assume that the figure formed by two points is not always superposable upon that which is formed by two other pts.; for otherwise any two lengths whatever would be equal to each other. Now this is an important property of any group.

I implicitly enunciate a similar hypothesis when I pronounce the word angle."

I have still to quote his ideas concerning contradictions in geometry as I think they are of cardinal importance. Here is what he says,

"In following up all the consequences of the different geometrical axioms, are we never led to contradictions?... The axioms are conventions. Is it certain that all these conventions are compatible?

"These conventions, it is true, have all been suggested to us by experience but by crude experience. We discover that certain laws are approximately verified and we decompose the observed phenomenon conventionally into two others: a purely geometrical phenomenon which exactly obeys these laws; and a very minute disturbing phenomenon.

"Is it certain that this decomposition is always permissible? It is certain that these laws are approximately compatible for experience

shows that they are all approximately realized at one and the same time in nature. But is it certain that they would be compatible if they were absolutely rigorous?"

"For us the question is no longer doubtful. Analytical geometry has been securely established, and all the axioms have been introduced into the equations which serve as its point of departure; we could not have written these equations if the axioms had been contradictory. Now that the equations are written they can be combined in all possible manners; analysis is the guarantee that contradictions shall not be introduced."

We see thus that both Lie and Poincaré are of the opinion that the question about the foundations of geometry represents a more concrete and, therefore, more easily manageable problem, than Klein and Killing and some others are willing to grant. Neither of the former mathematicians

allows an infinity of contradictory geometries, and Poincaré does gives some reasons for our choice in favor of the Euclidean geometry. He only thinks that this is solely due to the mode of experience we have of space, and when he speaks of hypothetical beings, whose experience might have led them to a predilection for the geometry of Lobatchevski, he certainly is right, in the sense that our geometrical notions, such as they are, are not altogether independent of the mode of experience we have and the nature of the universe we live in. In Poincaré's own words,

"It is our mind that furnishes the category for nature. But this category is not a bed of Procrustes into which we violently force nature, mutilating her as our needs require. We offer to nature a choice of beds among which we choose the couch best suited to her stature."

Now this view of the matter is certainly

more encouraging to the one who would venture to find, by methods more elementary and, consequently, more legitimate ones for the given purpose, than those based upon the laws of continuous groups, exactly which couch is the most suitable for nature's stature and upon what conditions the other couches may become just as suitable.

Let us now turn to Riemann whose paper on the foundations of geometry seems to have been the occasion apparently unintended by the author*) of many a misconception sanctioned by his name. Riemann himself, as well as can be gathered from the fact that he tried to find the laws of free mobility in manifolds of n dimensions, considering space as a special case of such manifolds where $n=3$, seems to have been of the opinion that geometry as a science of space and special magnitudes alone, must be one and only one, although he did not decide in favor of any of the three possible systems. He thought, at

*) See above, note to p. 7.

any rate that the necessity of the truth concerning the nature of the geometry of our space presents a concrete and not unsolvable scientific problem. He stated expressly that the solution of this problem was not to be found in investigations of such a general character, as was his own about manifolds in general a thing which his followers have not always heeded sufficiently. Since space was for him a manifold of special character, whose science alone he called geometry, as distinguished from the science of the general laws of manifolds, which, according to him belongs to analysis and the theory of functions. So he says on p. 258 of his "Math. Werke": "These magnitude relations admit of investigation only in terms of distinct quantity, their mutual connection being representable by formulae; under certain assumptions they can however be decomposed into relations which taken separately are capable of a geometrical representation, and through this it becomes possible to express geometrically the results of the calculation.

So that although an abstract investigation by means of formulae remains unavoidable, it will still be possible to invest its final results in geometrical attire". And on p. 256 he says, "More frequent occasions for creating and developing these conceptions of multiply-extended manifolds we find only in the higher mathematics." — His own investigation he considered useful only in so much as it threw light upon the extent and nature of the implicit assumptions of geometry and upon the many questions of measurement in the wider region of multiply-extended manifolds upon which these assumptions touch, and which, apparently, have escaped the attention of his predecessors. He says on p. 268, "Such investigations, which, like the one here carried through, start from general conceptions can serve only the end that this work (the investigation of the real facts underlying our notions of space and of its magnitudinal relations) shall not be hampered by too narrow conceptions and that the progress of discovery of

the connection of things should not be impeded by the burdens of inherited prejudices." — He looked, however, for the solution of this problem in the wrong direction, when he thought that some physical hypothesis which may in time prove necessary to account for certain, as yet unexplained physical phenomena in the realm of the infinitesimally small might also throw light upon the true nature of geometry. He certainly erred in respect of this physical hypothesis of the geometrical properties of our space*). They would lead to no better parallaxes results than astronomical observations upon stellars instituted with the purpose of finding some testimony in the immensely large triangles concerning the amount by which the sum of the three angles of a triangle is less than two right angles, as is very evident from the truly philosophical treatment of this subject by Poincaré. But be this as it may it still remains a fact that Riemann in the first place regarded space as being an unbounded manifold of three dimensions

*) "It must therefore, that either the reality which lies at the basis of space forms a discrete manifold or that the foundation of its magnitudinal relations ought to be looked for outside in the binding forces working upon it." (p. 268).

and spoke of it as being an empirical certainty greater than any other we have, and, secondly, thought that the problem as to the admissibility of the propositions of the Euclidean geometry beyond the bounds of observation was still an unsolved but not unsolvable scientific problem.

Some of the results arrived at by Riemann are:
1) The simplest form of the linear element in any **n-fold** extended manifold, admitting of measurement is $ds = \sqrt{\sum a_{ik}\, dx_i\, dx_k}$, where the a's are continuous functions of the x's; of these, n fns. can be taken arbitrarily, and $\frac{n(n-1)}{2}$ are fixed by the nature of the manifold. In space, for instance, even if it were curved, three of the a's could be taken = 0, each, and the rest would have to take their chances, which would depend upon the nature of the curvature. At this stage of his investigation Riemann seems to assume that space and the plane are flat manifolds in that

their linear elements can be brought to the form of $\sqrt{\Sigma dx^2}$ (s. p. 200 Werke). It would seem therefore, that when later he speaks of the possibility of space-curvature, and of a physical investigation in the realm of the infinitesimally small he means rather that since he does not see any logical a priori necessity of the necessary and sufficient assumptions of the Euclidian geometry which he established in §1 of art. III p. 205, he hopes to find an explanation of their necessity in physics, since he does not hope to find light on this subject in geometry, per his realm being only the finite —

2) In an n-manifold we can construct at each pt. ∞^{n-1} geodesics; then a surface-element is determined by any two of these given by their linear elements at the point and the surface is determined by two linear elements when these are prolonged until they become finite geodesics. In other words, Klein puts it in his "Nicht Euclid. Geom"

p 211, we have to consider the collectivity of geodesics whose linear elements $ds_i = \lambda' d'x_i + \lambda'' d''x_i$, or such whose initial directions are in the same linear manifold with the two given ones. Each of the surfaces thus obtained will have its own initial Gaussian curvature which Riemann defines as the curvature of the n-manifold at the given point in the given surface-direction

3) A manifold of constant curvature is such as has the Gaussian curvature in its surface-elements the same at all points and in all surface-directions. But the nature of the manifold at a point will be completely determined as soon as the surface-curvature is given in $\frac{n(n-1)}{2}$ surface-directions

4) Only a manifold of constant curvature allows free mobility of figures, and if the Gaussian curvature (of such a manifold) be denoted by α the linear element can be reduced to the form $\frac{1}{1+\frac{\alpha}{4}\Sigma x^2}\sqrt{\Sigma . dx^2}$.

5) All metrical relations of the manifold

depend upon the value of the curvature. The number of ways in which an n-manifold can move in itself without deformation is $\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}$, = number of coordinates − number of distances of n points.

Riemann then gives five possible forms of the conditions necessary and sufficient to determine the measure-relations of Space, as distinguished from all other three-dimensional manifolds admitting of measurements and flat in their smallest parts, i.e. such whose line-length is independent of position and in which the linear element is expressible as the square root of a positive differential expression of the second degree:

1° The Gaussian curvature in three surface-directions is zero at each point; or, otherwise, the metric relations of Space are completely determined if the sum of three angles of a triangle is always equal to two right angles.

2°. Besides the independence of line-lengths from position we may assume with Euclid the existence of rigid bodies independent of position, which is equivalent to postulating constant curvature. The sum of the three angles in all triangles is then determined when it is known in one triangle.

3°. We may assume not only the independence of line-length from position, but **also** the independence of length and direction of lines from position.

Each of these three alternatives adds something to the properties of a manifold, flat in its smallest portions. The first and last lead to the Euclidian geometry; the middle one allows the possibility of all three different systems, according as the sum of the angles of a triangle is greater, equal or less than two right angles.*) —

The position and the opinions of this second category of mathematicians and especially that of Lie, from

*) See Lie, Transformationsgruppen, vol. III p. 497, where he finds this paragraph in Riemann's (Ges. Werke, § 1 of Art. III) not clear.

to indicate that the task of establishing an efficient system of axioms is not perfectly hopeless. And if the reader admit with Lie that but a few investigations have materially furthered the process concerning the foundations of geometry, — he may find it of interest to read through the present memoir, even if its purpose is disclosed at the very beginning to be the establishment of such a system and the restoration of a goodly part of the old prestige to the origin and foundations of the geometrical science. —

It may not be amiss here, in this connection, to remind the generously disposed and impartial reader that this problem, besides its philosophic interest, is also of importance from a purely mathematical point of view. The fact which will substantiate my statement is, indeed, very well known to mathematicians who have even superficially familiarized themselves with the Non-Euclidean Geometry, although little stress is put upon its bearings by those

mathematical writers on the subject, who, having satisfied themselves that within the bounds of our limited experience the Euclidian geometry holds, concluded that beyond those limits actual deviations of the metrical relations of space may take place, of which we are not bound to take heed in our analytical geometry, as long as we intend only to avail ourselves of its results in actual practice. Already the earliest Non-Euclidians, and among them the two great founders of the hyperbolic geometry, Lobatchevski and Bolyai, have made it clear that the theory of proportion and similar figures is based upon the parabolic system of measurement, and that it has no meaning when the Euclidian postulate of parallels does not hold. In fact, they have both given formulae for the solution of rectilinear triangles, perfectly analogous to those of the spherical trigonometry.* Further, Bolyai has shown that by assuming the hyperbolic geometry

* See Lobatchevski, Theory of Parallels, translated by Halsted, pp. 35-45. Also Urkunden zur Gesch der Nicht Euklid. Geom., F. Engel, pp. 216-235.

to be true the problem of the squaring of the circle presents no difficulty* It is, therefore, evident that the establishment of the Euclidean geometry on a basis more rational than mere empiricism, even if very accurate, still remains a desideratum.

It will however become incumbent upon me to explain my own point of view in this matter and give in outline the results at which I have arrived and now to throw some light upon the methods pursued, as well as the reasons which according to the best of my judgment can be assigned to the final success with which these methods have been awarded.

To give my own views upon the role of experience and reason in the formation of our geometrical conceptions would, I think, only be a repetition of what is stated more or less explicitly in my introductory chapter on dimensions, as well as a repetition of many excellent remarks of Poincaré in his paper in

*) See "Science Absolute of Space" by John Bolyai Halsted's translation, p. 47.

the Monist which I have allowed myself to quote so extensively. In a few words these views may however be summarized thus:-

Rule in all pure sciences are fundamental conceptions, in Geometry formed by experience helped on by pure reasoning which abstracts from certain unessential irregularities in the rough data of experience by reducing certain general norms to ideal forms not admitting of exception. The exceptions indeed are purposely eliminated by ascribing to them some other causes which are not the subject of the given investigation So for instance, in mechanics the fact, that no body, in actual experience, possessing a certain momentum can go on and move forever does not bother the physicist who postulates the first law of Newton and ascribes the stopping of the body to the retardation of its motion to external causes, like frictional resistance etc. Similarly, if ideal solids are postulated in geometry, the deformation which these solids

undergo in motion is ascribed to physical causes and not to properties of space*)

Further, I think that we cannot start simply with an axiom-that space is a number-manifold, i.e. each point in it can be determined by three coordinates, or three numbers which can be made to vary continuously. I rather think that space ought to be proved capable of being made a number-manifold; and the best starting point in this direction is, according to my opinion, to be found in the simple physical facts of impenetrability rigidity and divisibility of bodies even in geometry being idealized. The bulk of a given portion of space bounded on all sides, can certainly be represented by a number showing how many times it will contain a smaller bulk of definite shape. By considering the smaller bulk as rigid, or such that no portion of it is allowed to move w. r. to the remaining portions of the same so that internal motion is excluded and making

*) See postulate 1), Definition 6 and scholium to Definition 8 of the introductory chapter.

this smaller bulk take up all possible positions within the larger bounded portion of space we observe that no matter where it be placed within the larger one, it always occupies or fills up the same numerical portion of the larger bulk: the number of other bulks like the smaller one (such that, if broken up into infinitely small portions, free to change position w.r.t. each other will always fill up completely the same space) necessary to fill up the larger one besides the smaller one itself, or the number of places the smaller, allowed to change its form in all possible ways so as to retain only impenetrability can be made to occupy within the larger, such that no two have any portion in common, is always the same*). We postulate that this be true for any bounded space and for any small bulk which is placed in the larger one in any position, that there should always be the same numerical relation between the smaller and the larger bulk, as soon as these are given as a rigid solid on the one hand, and a bounded vacuum in which the first is to lie in any position, on the other hand.

*) For a complete and rigorous treatment of this question the reader is referred to the Introductory Chapter of my Dissertation Scholium to Definition 8). There is only possible a short indication of the procedure.

We arrive at the notion of congruent portions within the bounded space meaning such which the same solid fills up to the exclusion of others, and by considering besides very small portions of the smaller bulk, their number and disposition w. r. to one another is seen not to change as long as rigidity of form is postulated for the whole. So that after we have worked ourselves up to the notion of surface, line and point, as done in the introductory chapter, rigidity is seen to imply that no two points separated in one position of a rigid body become ever coincident on account of change of position of the solid. And, moreover, the same continuous series of separated pts. of the solid must be capable of being construed between any two given points of the solid in any one of its positions as in any other.

But this is only a starting point. We must further make clear to ourselves what we understand when we say that space is a three-dimensional

manifold, considering a point as its element, and whether there is any sense in looking for a fourth dimension not directly given by experience. It will appear from the treatment of the question in the introductory chapter that the tri-dimensionality of space is actually postulated by the definition of a point and that, therefore to look for a fourth dimension, without changing its element from that which lies at the basis of the metrical geometry of Euclid to some other geometrical object which is, in fact, a figure in the Euclidian sense depending upon a certain number of parameters) as is done, for instance, in Plucker's line-geometry — is a contradiction in terms. Finally, from the same principle of rigidity the notion of distance, as an invariable relation between two points in rigid connection or in fixed space, is easily derived by a definition which makes use of the principle of superposition.

Next continuity must be postulated*, and then

*) See Axiom 1) p.

we must show how distances can be added and subtracted, and whether there is a line or a one-dimensional manifold in space, capable of representing by the actual distances of its points all possible distances arrived at by addition and subtraction and whether this can be done in a unique way. It appears that from this property alone the str. line can be deduced which will have all other properties of the str. line commonly postulated for it in the Euclidian geometry; the construction moreover, based upon this property, will make it a number-manifold*) of infinite extent, such as can in no way be mixed up with a geodesic, returning into itself at a finite distance. This, in a certain way disposes of the so called elliptic geometry of the str. line. Further the line so constructed will prove to be both an axis of rotation and an axis of helicoidal motion or in Poincaré and Lie's language, admitting either the transformations of a rotative sheaf, or of a helicoidal group according to

*) The construction makes the str. line a number-manifold in the Cantor sense, since as we can consider all possible sums of all possible rational numbers, we can construct the irrational numbers by sequences, in the way it is done by Cantor for pure nos.

one point at least upon the line is taken as an invariant point, or none at a finite distance is taken as an invariant pt., and the line is conceived to slide upon itself, all points in rigid connection with the line, but outside it, being free only to twist, i.e. to move with a screw-like motion.

The notion of distance, as here defined, is, of course, of a very abstract nature, and does not depend upon the paths which two non-coincident pts. can be made to pass from the position of the first to the position of the second in a given surface, nor upon any given position of the point-couple in space, but only upon the relative position of the two points themselves. It is simply a fact of experience that a pair of points in a solid are capable of coincidence only with certain determinate couples of points in other solids or vacant space and this fact of congruence or non-congruence is the only factor of deter-

mining equality or non-equality of distances. It is only after it has been proved that this geometrical magnitude is representable uniquely and perfectly by some line (a priori, a surface or a volume might perhaps have been found more adequate to represent this magnitude, as it happens for instance with the angular magnitude, which is equally well represented by a portion of a circular arc as by the area of a sector of the circle of radius unity and as for inst. the solid angle considered as a geometrical magnitude may be equally measured by the area of a spherical surface of radius unity, whose centre is the vertex of the angle, or by the corresponding spherical sector; so that it is only an accident, having of course its reasons in the nature of things a posteriori that distance as a geometrical magnitude has for its representation a line), it is only after this fact has been established, that any curve can be broken up into linear elements ds, each of which is comparable with the

elements of three given straight lines dx, dy, dz, and can be expressed in terms of these. So that any complicated expression for a linear element of some curve in space in terms of dx, dy, dz, say, $ds = f(dx, dy, dz)$ must necessarily be based, in the first instance, upon a certain relation, which could be considered the simplest and which would be formed directly in the same way as a finite distance is expressed in terms of three other finite distances, which must be taken as parameters in case of three-dimensional space.

I wish here to call attention to the fact that the deduction of the existence of the straight line in space, not as a line in the plane, as far as I am aware, seems never to have constituted a serious problem with mathematicians. This is perhaps the reason why the straight line has always been regarded as a geodesic which is determined by

two points in the surface to which it belongs, i.e. as a geodesic in some plane. The plane is postulated or constructed before the straight line and angles and triangles and circles are regarded not only as plane figures that is such as can lie in a plane, but also as figures constructed in a plane. The distinction, according to my mind, is not at all trivial, since the existence of a plane can be proved with perfect rigor only after a number of theorems concerning angles triangles and circles have been established for these figures in space. Then only, according to my opinion we ought to prove that these simple figures, as well as the straight line are plane figures.

The plane as constructed from a certain origin must then be shown to be capable of moving upon itself in a triply infinite number of ways and also of coincidence with itself when its two sides are interchanged. The first will establish the

legitimacy of what is called in analytic geometry change of origin and change of axes, the second a revolution of the plane through ~~through~~ an angle π, which appears in the theory of groups to need special subgroups of displacements, (See Lie Continuierliche Gruppen, p. 101. The first kind of displacements form the group of congruent figures, the second the group of symmetrical figures)*.

In the Euclidian geometry both processes are invariably used in superposition of figures for demonstrations. In spherical and pseudospherical geometry this is also practised with the understanding that bending without stretching is postulated. In spherical geometry bending is necessary only for making the inner side of a portion of a spherical surface coincide with the outer side of the same surface, or for the purpose of applying its figures to figures formed upon other surfaces of equal constant curvature; in case of pseudospherical

*) Groups of congruent figs, — $x_1 = x\cos\alpha - y\sin\alpha + a$; $y_1 = x\sin\alpha + y\cos\alpha + b$
Group of symmetrical figs, — $x_1 = x\cos\alpha + y\sin\alpha + a$; $y_1 = x\sin\alpha - y\cos\alpha + b$,
Ibid.

surfaces, i.e. surfaces of constant negative curvature no superposition of parts of different regions even on the same side of the same surface would be possible without bending. And it was just by abstracting from rigidity, in so much as bending without stretching was allowed, that Beltrami in his "Saggio"*) and in his "Teoria Fondamentale" was able to prove that Lobatchewski's geometry holds good in our Euclidean space upon surfaces of constant negative curvature which he first named pseudospherical surfaces. "The fundamental criterion of the demonstrations in the elementary (Euclidean) geometry" Beltrami begins his investigation in his Saggio "consists in superposition of figures. This criterion is applicable not only to the plane, but also to all surfaces upon which there can exist equal figures in different positions, that is to say to all surfaces where any portion can by means of simple flexion be applied to any other portion of the same surface.

*) "Saggio di Interpretazione della Geometria non-Euclidea". Giornale di Matematiche, 1868, t. VI. (See Note above, p).

We see, in fact, that the rigidity of the surfaces upon which the figures are traced is not an essential condition for the application of this criterion, for instance the exactitude of the plane Euclidean geometry would not become deteriorated, if we should begin by conceiving the figures traced upon the surface of a cylinder or a cone instead of a plane.

Stating then that the surfaces whose figures have a structure independent of position, and hence allowing the principle of superposition without restriction are those of constant curvature only, he goes on to say:

"The most important element of figures is the straight line. The specific characteristic of this line is that it is completely determined by two of its points, so that 2 str. lines cannot pass through 2 points in space without coinciding in all their extent. In plane geometry, however, this principle is used only in the following form:

In making coincide two planes . . . one of which there is a straight line it is sufficient that the two lines coincide in two points in order that they coincide in the whole of their extent."

"Now, this property the plane has in common with all surfaces of constant curvature, where instead of the straight lines we take the geodesics. . . If we make coincide two surfaces of constant and equal curvature so that two of their geodesic lines have two points in common these lines will coincide in all their extent

"It follows that, while the cases where this property is subjected to exceptions, the theorems of planimetry which are proved by the principle of superposition and of the postulate of the straight line for plane-figures are also true for figures traced in an analogous way upon surfaces of constant curvature by geodesic lines.

"Upon this are based the many analogies

between the geometry on a plane and that on a sphere, the straight lines corresponding to the geodesics, i.e. to arcs of great circles. For a sphere, however, there exist exceptions, for any two points diametrically opposite or antipodal points do not determine a geodesic without ambiguity since through such points an infinity of great circles will pass. This a reason why certain theorems in plane geometry are not true for the sphere, as for instance the theorem that two perpendiculars to the same line do not meet."

Beltrami further makes clear that the basis of investigation in plane geometry is too general, for, as usually done the only facts lying at this base are the principle of superposition and the postulate of the straight line. The results of the demonstrations must exist wherever this principle and this postulate are true. They must evidently be true for surfaces of constant curvature in which the postulate of the straight line holds without exception.

Now the purpose of Beltrami's investigation was precisely to show that this postulate does not admit of exception in case of surfaces of constant negative curvature. And in his own words, "If we can prove that such exceptions do not exist for these surfaces it becomes evident that the theorems of the Non-Euclidian planimetry hold without restriction upon such surfaces. And then certain results which seem incompatible with the hypothesis of a plane may become conceivable upon such a surface and obtain thereby an explanation not less simple than satisfactory. At the same time the determinations which produce the transition from the Non-Euclidian to the Euclidian planimetry are shown to be identical with those which specify the surfaces of zero-curvature in the series of surfaces of constant negative curvature".

Lobatchevski and Bolyai, who together with Legendre, were aware of the defects of the Euclidian geometry in respect of all the postulates regarding

the straight line and the plane, have both tried to construct them and to deduce their properties from the construction.*) But, as it seems to me, they did not succeed in doing it with sufficient rigor; and besides, neither of them freed himself from the idea that a plane is given in conception previous to a straight line and, therefore, they constructed first the plane and then the straight line in it. The straight line therefore, again has the same properties as a geodesic upon a surface which will coincide with another geodesic in a similar surface of same constant curvature, as soon as the two surfaces are superposed so that two congruent pairs of pts. in the two geodesics are made to coincide. At least, neither of the two mathematicians separated the straight line from the plane sufficiently to come to the clear idea that figures of straight lines in space can be considered without considering the planes in which they lie. This is one of the reasons why they could not prove the parallel postulate which, in fact,

*) See "Urkunden zur Geschichte der Nichteuklidischen Geometrie" by F Engel, 1898. Vasilej Iwanowitsch Lobatschefskij (pp. 93-109) and Frischauf, Elemente der Absoluten Geometrie. 1876 pp.

distinguish the plane from all other surfaces of constant curvature. For, it is evident that since the curvature of a surface is an extrinsic property of the surface i.e. it is a parameter by varying which we can obtain all surfaces of constant positive and negative curvature (i.e. continuously between the two being the plane (of zero curvature)*) — those properties of the geodesics which depend upon any particular value of the curvature could not be found, except by leaving the surface and going out into space. Of course, the expression of the linear element contains, give the true metrical properties of the corresponding surface, and hence also of the plane. But this very expression, as Riemann has shown, depends upon the curvature (In fact, for a surface of Gaussian curvature $\frac{1}{K^2} = \alpha$, the linear element is reducible to the form $\frac{1}{1 + \frac{\alpha}{4} \sum x^2} \sqrt{\sum dx^2}$. so that in case of positive curvature α is + of negative curvature α is — and in case of the plane, if the 'parallel postulate' holds, $\alpha = 0$ and conversely, if $\alpha = 0$ then the parallel postulate holds).

*) $\pm \frac{1}{K^2}$ gives curvatures of spheres and pseudospheres with all their varieties resulting from bending, and $K = \infty$ gives the plane, See Riemann Ueber die Hypothesen etc. II, 5.

Hence a line is a minimum in the surface itself and supposing we do not know the form of the linear element, we could in no way prove or disprove the parallel postulate. And this is exactly the reasoning of those who think that the parallel postulate cannot be proved.

I could not do better than to refer again to the last quotations from Beltrami who although he himself as far as I can see from his "Saggio" and "Teoria Fondamentale" may have never expressed himself directly on the possibility or impossibility of proving the parallel postulate, showed, however, the reason why Lobatchevski and Bolyai had arrived at a geometry for the plane which is actually true only for pseudospheres.

In another place in his "Saggio" he says: "We see, then, that two points of the surface (pseudospherical surface), chosen in any manner whatever always determine uniquely a geodesic line... Thus the surfaces of constant negative curvature are not subject to exceptions which in this respect happen in the case

of surfaces of positive curvature and, therefore, we can apply to them the theorems of the Non-Euclidean planimetry. "However, these theorems in their greatest part are not susceptible of a concrete interpretation except if they are referred precisely to these surfaces instead of the plane, as we are going to prove presently in detail."

It is interesting to compare ~~with~~ this sober view upon the Non-Euclidian geometry of the great Italian mathematician, whose works more than those of any other mathematician succeeded in putting this geometry upon a respectable footing, with a few quotations from Bianchi, "Lezioni di geometria differenziale", German translation, 1898, t. II, p. 434. The quotations referred to are headed by the superscription,—

"A review of the Non-Euclidian Geometry" and follows an excellent account of pseudospherical geometry treated by the method of conform representation, and they read as follows.

"In the principal theorems of the pseudospherical geometry which we have deduced in the preceding paragraphs a close analogy is observable to the propositions of the plane and spherical geometry. The basis for these analogies, as well as for the differences we can foresee a priori. If, in fact, we examine the axioms and fundamental postulates of the plane geometry, as laid down by Euclid in his first book, and if in case of pseudospherical surfaces we replace the str. line by the geodesic line we see that when we leave out of consideration the XII postulate concerning parallels all others will hold without change in the pseudospherical geometry. This is, in particular, the case with the principle of congruence of figures and also with the principle that a geodesic line is uniquely determined by two of its points. Those propositions of plane geometry, which do not depend upon the parallel postulate, hold, therefore, for the pseudospher- geometry.

the ∞ is undergo such changes as to become identical with the old ones [illegible] the radius R of the pseudo spherical surface is made infinite.

The above considerations show readily the impossibility of all attempts made to prove the parallel postulate. If this proof could be logically deduced from the other principles, it would have equally to hold for the pseudospherical surfaces in Euclidian space."

"And, in fact, if in the plane geometry we drop the Euclidian postulate, we are led to the so called abstract or Non-Euclidian geometry whose foundations were laid down by Bolyai and Lobatchevski, and which (the straight line being taken as unlimited) perfectly coincides with the pseudospherical geometry."

Now this objection is certainly valid if we were bound to consider only plane figures. Fortunately, we can have constructions involving distances,

or straight lines and angles not bound (to be in any particular surface at all. The conception of a quadrilateral without fixed area consisting of four fixed distances, which are equal by pairs (opposite sides) and whose angles are variable and equal by pairs, is an important conception in elementary geometry. A triangle is fixed as soon as its sides are fixed: it cannot 'rack'; its area and its angles are fixed with its sides; it is necessarily a plane figure. And if the proposition is true that the area of a geodesic triangle = const. times the excess or deficiency of the sum of the angles of the triangle over π, i.e. $\Delta = \pm K^2 (A+B+C-\pi)$, where Δ is always positive and K is the radius of curvature, we have no means of varying this area, when the geodesic triangle is conceived as bound up with the surface. But*) the 'immaterial quadrilateral', which consists always of two triangles, is not bound up with any fixed area and is not bound to move in any particular

*) See Definition 10) of my Dissertation.

surface, and its angles, under certain imposed conditions, being in certain determinate relations among themselves are still variable, and by considering the continuous series of deformations given in Theorem 16) we arrive at the conclusion that there must exist a quadrilateral with equal opposite sides, whose two angles adjacent to the same side are equal 2 rt. Δ's, not knowing however whether the quadrilateral is a plane quadrilateral or not. And then only with the aid of the lemma (Theorem 17), which proves that the sum of any two face angles of a trihedral angle is greater than the third, we find either that such a quadrilateral must be a plane quadrilateral or that the sum of the three angles is > 2 rt. Δ's. But the second alternative, which holds for surfaces of positive curvature was proved to be impossible in the case of a plane by Theorems 12 - 15; hence only the first alternative remains from which it immediately follows that the sum of the three angles of a rectilinear triangle

is equal to two right angles, or in other words the geometry of the plane is parabolic.

Now it so happens that this proof has been framed in such a manner as to take in consideration the existence of both the spherical and the pseudospherical geometries. It actually leaves room for the existence of surfaces of constant curvature where the sum of the three angles of a geodesic triangle is greater than 2 rt. angles, and such where this sum is less than 2 rt. angles. The first prove to be finite in extent — which follows immediately from theorems 12 & 15, since these theorems assume the impossibility for two geodesics in a plane to intersect in two points and the infinite extent of the plane (both of which assumptions have been proved in theorems 2, 3 & 6) — and all their normals meeting in a point at a finite distance; while the second class of surfaces have their normals non-coplanar, or meeting at an imaginary point. Hence, the plane appears to be the limiting case between the two.

being infinite in extent and having its normals meeting in a point at ∞. The considerations of Bianchi and others which they put forward against the possibility of proving the Euclidean postulate on the ground that if it could be proved for the plane it would have to hold for the pseudosphere (or for the sphere), evidently have no value against such a proof where both of these cases have been considered, and where the proof first establishes the existence of a kind of metrics somewhere in space, which afterwards is proved to be possible only for the surface of zero curvature, which is the plane. Then only follows the proof of Euclid's postulate proper for the plane which by its metrics is now singled out among all surfaces of constant curvature, as the only one for which the sum of the angles of a triangle equals 2 rt. ∠s.

If we examine in detail the axioms which have been made use of in my Dissertation, we find that they are not only those which are necessary for defining

the group of continuous motions in general, without, however, postulating that a point in space is definable by three coordinates. 2d the postulates that Helmholtz enumerates in his work "Ueber die Thatsachen, die der Geometrie zum Grunde liegen" Kön. Ges. der Wis. zu Göttingen, 1868 I assume only the first and the third stated as follows:

1) Continuity in motion defined exactly in Axiom 1 on p. of my memoir.

2) Free mobility of rigid bodies and consequent independence from position of spacial figures of all kinds.

To these, whose order, of course, is reversed, I add impenetrability and infinite divisibility. The number of dimensions I do not assume, but I deduce from the postulates.

A word still with regard to the Non-Euclidian geometry. What is its significance and the place it

has to occupy in the general science of geometry?

I shall here, even at the risk of repetition, recapitulate and supplement in a concise way what I have said with reference to the two Non-Euclidean groups of motions obtained by Lie from the Riemann Helmholtz axioms, as modified by him. I think, namely, that in this regard the position of Beltrami in his "Saggio" and "Teoria Fondamentale" is, in general, the only tenable one. In our point-space of three dimensions the Lobatchevskian and Riemanian geometries, for two dimensions in space, are realized respectively on the pseudospherical, and surfaces of constant positive curvature, and they can have no other concrete interpretation. With special conventions, however, as to the meaning of 'distance' and 'angle' we get the Poincaré interpretation by means of what he calls the quadratic geometries, i.e. geometries on certain quadric surfaces. Professor Klein's interpretation of the generalized metrics can in no way make the

plane become either an elliptic or an hyperbolic
and if Cayley's generalized metrics can be turned into
account for obtaining the elliptic, hyperbolic and
parabolic geometries, it is only for different sur-
faces actually existing in the Euclidean space that
this interpretation can be of any value. The very arbit-
rariness of these different kinds of metrics, which
depends upon the arbitrary value of the constant
C in the formula*) $(x,y) = C \log[x,y,O,O']$, where O,
O' are the points of intersection of the range xy with
the fundamental quadric, shows that the different
metrics must refer to a whole series of different
two-dimensional manifolds, differing in curvature
and constituting the elements M_2 of some K_2. The
aggregate of all these however, will in our case, constitute
a flat manifold of three dimensions, namely the
point-space of our experience - just like the aggregate
of all possible plane curves of different curvature
passing through a point in a plane, constitute a plane

*). Math. Ann. Bd 6.

manifold of two dimensions.

As to the Non-Euclidian metrics in three dimensions, I cannot see any interpretation for this, unless the space to which these metrics refer, be a derivative manifold contained in a higher manifold of four dimensions, since again the very parameter of the curvature suggests only a particular case out of an infinity of possibilities, arrived at by giving the parameter all possible values ranging from $-\infty$ to $+\infty$ and the aggregate of all these would then fairly make a manifold of four dimensions. Here, again, Beltrami seems to have hit the truth with regard to this interpretation of the geometries of Riemann and Lobatchevski for space of three dimensions.

Thus at the beginning of his "Saggio" Beltrami says:

"We have sought to render account to ourselves of the results to which this new doctrine (the geometry of Lobatchevski) leads; and following a process which seemed to us actually

to conform to the good tradition of scientific investigation, we have tried to find a real basis to these results. We think we have found one for the planimetric part, but it seems to us impossible to find one for the case of three dimensions. - And then at the end of the same work after having proved the interpretability of Lobatchevski's planimetry in every particular by the geometry upon pseudospherical surfaces, Beltrami goes on to say: "From the very nature of the interpretation, we can easily foresee that there can exist no analogous interpretation, equally real, for the Non Euclidian Stereometry. In fact, in order to obtain the interpretation which we have just given, it was necessary to substitute for the plane a surface which cannot be reduced to a plane, that is to say, whose linear element can in no way be reduced to the form $\sqrt{dx^2+dy^2}$ which essentially characterizes the plane itself. Consequently if we were lacking the notion of surfaces non-applicable to a plane, it would be impossible for us to attribute a

visible geometric meaning to the constructions which we have developed up to this point. Now the analogy leads naturally to think that if there can exist a similar interpretation for the Non Euclidean tercometry, this interpretation must be deducible from the consideration of a space whose linear element is not reducible to the form $\sqrt{dx^2+dy^2+dz^2}$ a form which essentially characterizes the Euclidian space. And since up till now, as it seems to us, we have been missing the notion of a space different from the Euclidian or, at least, such a space is beyond the domain of ordinary geometry, it is reasonable to suppose, that even if the analytic considerations, upon which the preceding constructions are based, were susceptible of being extended and carried over from the region of two variables into that of three variables, nevertheless the results obtained in this last case could not be interpreted by the ordinary geometry. This conjecture requires a degree of probability bounding very closely on certainty,

when one undertakes to extend the preceding analysis to the case of three variables. —

Putting

$$(18') \quad \left\{ \begin{aligned} ds^2 = \frac{R^2}{(a^2-t^2-u^2-v^2)^2}\Big[& (a^2-u^2-v^2)dt^2 + (a^2-v^2-t^2)du^2 \\ & + (a^2-t^2-u^2)dv^2 + 2uv\,du\,dv + 2vt\,dv\,dt + 2tu\,dt\,du \Big], \end{aligned} \right.$$

which takes the place of (1) in two dimensions*), — it is easy to assure oneself that the analytic deductions obtained from formula (1) subsist integrally for the new expression, and that the value of ds given by this last is effectively the value of the linear element of a space in which the Non-Euclidian (Lobatchewskian) Stereometry finds an interpretation as complete, resulting analytically as that given for the planimetry."

"By" putting

$$t = r\cos\rho_1, \quad u = r\sin\rho_1\cos\rho_2, \quad v = r\sin\rho_1\sin\rho_2 \text{ and}$$

$$\frac{R\,a\,dr}{a^2-r^2} = d\rho, \quad \text{we get}$$

$$ds^2 = d\rho^2 + \left(R\sinh\frac{\rho}{R}\right)^2\left(d\rho_1^2 + \sin^2\rho_1\,d\rho_2^2\right),$$

a formula which shows that ρ, ρ_1, ρ_2 are the orthogonal curvilinear coordinates of the space considered."

*) The formula (1) here referred to is the one which Beltrami gives at the beginning of his work for the linear element of a surface of constant negative curvature, $ds^2 = R^2\frac{(a^2-v^2)\,du^2 + 2uv\,du\,dv + (a^2-u^2)\,dv^2}{(a^2-u^2-v^2)^2}$; this serves as the basis for his interpretation.

Now, M. Lamé has proved that taking as curvilinear coordinates of points in space the parameters ρ, ρ_1, ρ_2 of three families of orthogonal surfaces - in which case the distance between two infinitely near points is represented by an expression of the form

$$ds^2 = H^2 d\rho^2 + H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 -$$

the three H's as functions of the ρ's must satisfy two distinct systems, each consisting of three partial differential equations of the types,

$$\frac{\partial^2 H}{\partial \rho_1 \partial \rho_2} = \frac{1}{H_1} \frac{\partial H}{\partial \rho_1} \frac{\partial H_1}{\partial \rho_2} + \frac{1}{H_2} \frac{\partial H}{\partial \rho_2} \frac{\partial H_2}{\partial \rho_1} \quad \text{and}$$

$$\frac{\partial}{\partial \rho_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial \rho_1}\right) + \frac{\partial}{\partial \rho_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial \rho_2}\right) + \frac{1}{H^2}\frac{\partial H_1}{\partial \rho}\frac{\partial H_2}{\partial \rho} = 0$$

(Leçons sur les Coordinées curvilignes, p. 76 & 78).

In our case

$$H = 1, \quad H_1 = R \sinh \frac{\rho}{R}, \quad H_2 = R \sinh \frac{\rho}{R} \sin \rho_1,$$

and for these values the first system is identically satisfied; but the second system is satisfied only for $\rho = \infty$. Hence the expression (18) cannot belong to the linear element of the ordinary Euclidian space, and the formulae founded upon this expression

cannot be constructed by means of figures ~~[illegible]~~ as by the ordinary geometry."

And again in his "Teoria fondamentale degli Spatii di Curvatura constante" Beltrami says:

"Thus, all conceptions of the Non-Euclidian Geometry find a perfect correspondence in the geometry of a space of constant negative curvature. It is only necessary to observe that while the conceptions of the planimetry obtain a true and proper interpretation since they can be constructed upon a real surface, those, on the contrary, which refer to three dimensions are susceptible only of an analytic representation, since the space in which such a representation could be realized is different from that to which we ordinarily give the name of space. At least it does not seem that experience could be brought in agreement with the results of this more general geometry, unless we suppose R to be infinitely great, that is, the curvature of the space to be zero. Which circum-

stance might, of course, also be due only to the smallness of the triangles which we can measure or to the smallness of the region to which our observations extend themselves."

In his Teoria Fondamentale Beltrami shows from a general discussion of n-dimensional manifolds, that the linear element in the Riemannian geometry of three dimensions may be taken to be the same as the linear element upon a hypersphere in a space of four dimensions. The equation of the hypersphere with centre at origin will be $t^2+u^2+v^2+w^2=a^2$ and hence

$$d\sigma^2 = dt^2+du^2+dv^2+dw^2$$

is at once the representation of the linear element upon the hypersphere of radius a and a Riemannian space of curvature $\frac{1}{a^2}$.

To obtain the linear element of the three dimensional space of Lobatchewski, he substitutes $ds = -\frac{K\,d\sigma}{w}$ and by eliminating w he gets (18). The curved Lobatchewski space of infinite extent is

this imaged upon the interior of the sphere of Euclidean space, $t^2+u^2+v^2=a^2$, — the geodesics of that space being represented by chords of the sphere. Every geodesic has two distinct real points at ∞ which are imaged upon the representative sphere by the 2 ends of the corresponding chord so that the spherical surface itself corresponds to the [illegible] of all points at ∞ in the Lobatchevski space. —

But since we have proved that to assume a four dimensional point-space is to commit a logical error, and since Beltrami's results have certainly given a conclusive analytical proof that we could obtain Lobatchevski's geometry for three dimensions, if we could actually construct a curved three-dimensional manifold contained in a four-dimensional plane-manifold — we may surmise that the only way to obtain a concrete and true interpretation of the Lobatchevskian (as well as the Riemannian) stereometry is to be found in Plücker's idea that our space becomes

a manifold of a higher number of dimensions, when, instead of the point we take as its element a figure depending upon n parameters, making space a manifold of n dimensions. Therefore, it would seem that one of the simplest ways to look for such a concrete interpretation would be to start with line geometry which makes space an R_4, the next simplest after the point-space which is an R_3, and seek what in this geometry would be meant by the terms distance angle linear element, curvature, parallel, perpendicular and other metrical terms, and see whether the results arrived at by considering in it the special three-dimensional manifolds (complexes) possessed of "curvature" (since, by excluding the postulate of parallels, which is now proved for a flat manifold as our point-space must undoubtedly be) we actually obtain manifolds of "constant curvature" to agree with those obtained in the Lobatchevskian and Riemannian geometries, respectively. But such an investigation would go already far beyond the limits of the present Dissertation.

Autobiography.

The author of this Dissertation, Israel Euclid Rabinovitch, was born in the month of June of the year 1861, in the town of Berdichev, Kiev Government, Russia. He obtained his elementary education in an old-fashioned rabbinical school, and afterwards prepared himself by self-instruction for the entrance into a Russian University, by having completed a course of a Russian classical Gymnasium. Having, however, found it difficult to obtain admission there, he left in 1887 for the United States, with the express purpose of eventually pursuing an academic course of studies in one of the American Universities. He entered the University of Pennsylvania in 1890 and, holding a scholarship there, pursued the elements of mathematics, including some engineering courses, under Professors Crawley, Fisher, Barker and Spangler, during 1890-91 and part of 1891-92, when he left on account of illness. During the academic

year of 1894-95 he pursued special courses of mathematics and German in Harvard University. In 1896 he entered the Johns Hopkins University, as a student of advanced standing, and has since been pursuing courses in Mathematics under Professors Morley, Craig, Cherriman and Hulburt and Dr Cohen and Physics under Professor Ames and Dr Bliss and Philosophy under Professor Griffin. He was made a candidate for the degree of Doctor of Philosophy in 1899, having selected Mathematics as his principal subject and Physics and History of Philosophy as first and second subordinates, - and was appointed Fellow in Mathematics in June 1900. The special research work connected with the review of the Non-Euclidian Geometry, as given in the introduction to the present Dissertation, has been done during the present year (1900-1901) under the direction of Professor Morley.